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MEASUREMENT-THEORETIC OBSERVATIONS ON FIELD'S INSTRUMENTALISM AND THE APPLICABILITY OF MATHEMATICS

Davide Rizza

Abstract
In this paper I examine Field's account of the applicability of mathematics from a measurement-theoretic perspective. Within this context, I object to Field's instrumentalism, arguing that it depends on an incomplete analysis of applicability. I show in particular that, once the missing piece of analysis is provided, the role played by numerical entities in basic empirical theories must be revised: such revision implies that instrumentalism should be rejected and mathematical entities be regarded not merely as useful tools but also as conceptual schemata by means of which we can articulate our understanding of experience.

1. Introduction

In *Science without Numbers* (1980) Hartry Field argues against that form of realism about mathematical entities based on the role they play within scientific theories. In so doing, he develops an account of classical mechanics which dispenses with the usual calculus tools adopted in physics.

To my knowledge, the philosophical debate on Field's book has mostly been restricted to the ontological implications of his position and the problems raised by the kind logic (second-order) adopted in his treatment of mechanics. There is, nevertheless, another important issue connected to it, and that doesn't seem to have received too much attention, namely that of the applicability of mathematics to physical (in general, empirical) facts.

More precisely, there is a natural way of reading Field's results as an extremely interesting analysis of the applicability of mathematical entities to scientific theories: this is because, in order to explain why we can dispense with the calculus in mechanics, Field shows how the real numbers, together with functions of one or more real variables and their properties (like completeness, continuity or differentiability) can be introduced on a synthetic theory of space-time, that only contains primitives (e.g. space-time points and the geometrical relations of order and congruence) with a straightforward physical interpretation, which are, for this reason, opposed to abstract entities, relations and operations. Within this framework it is possible to explain, for example, what's the intrinsic content of an analytical operation like differentiation, by isolating the physical notions that allow its introduction on a coordinatized counterpart of space-time.

What we are able to tell is, in short, what physical conditions must certain entities, like space-time points, satisfy in order for them to be describable numerically: we formulate explicitly those physical facts entail the possibility of superimposing on the structure of space-time the structure of the vector space R^4 (whose elements are all the ordered 4-tuples of real numbers).

This act of superimposition generates a distinction between, on the one hand, physical entities (e.g. space-time points), that we can treat independently and, on the other, numerical entities (e.g. the reals), that are available to us as tools to facilitate deductions about the physical entities. Physical entities are of course indispensable for the description of space-time, while numerical entities are regarded as extrinsic to it: the resulting picture of mathematics is therefore instrumentalist, as it restricts its role to that of a useful device for drawing more expediently inferences from (numerically interpreted) physical premises.

In this paper I want to argue against this conclusion: in particular I make a case for the idea that theoretical entities (like space-time points) and mathematical ones are similar in one important respect, i.e. in that they are schemata employed to embed experience into an ideal model. As I'll show, this doesn't conflict with the account Field gives of applicability but rather supplements it. The position I want to maintain is aimed against Field's distinction between the intrinsic[1] features of theoretical entities and the extrinsic features of numerical entities: instrumentalism is a consequence of extrinsicity or, equivalently, dispensability.

On the view I'm proposing, on the other hand, it is possible to explain the role of real numbers in applications also in terms of their 'intrinsic' content (in a sense to be explained). Since I take instrumentalism to follow from the extrinsicity of mathematical entities to the description of physical settings, I block it by showing that extrinsicity is only tenable on the basis of a partial account of applicability which, when extended, proves incompatible with instrumentalism.

The main consequence of my position is that mathematical entities may be understood not just as useful tools but also as ideal schemata that deepen our understanding of facts[2].

[1] This adjective is taken here and in any subsequent occurrence in its mathematical meaning, which is 'invariant under a class of coordinate changes'.

[2] I don't think such a conclusion needs any realist attitude regarding mathematical entities: on the contrary, it seems to me that, as long as we talk of ideal schemata, we should abandon even a realist attitude toward theoretical entities. This move does not, in my opinion, imply fictionalism, at least a strong version of fictionalism, that sees theoretical entities as, again, useful tools for drawing conclusions from experience. Firstly, because we need to explain why certain postulated entities are useful and this doesn't in general reduce to mere convenience, and secondly because it appears to me the alternative between realism and strong fictionalism (perhaps fictionalism in general) is a false one. It appears more reasonable to think of our theories invoking ideal

In sum, this paper is structured as follows: in Section 2, I introduce an account of the applicability of real numbers to measurement, that I show to contain all the conceptual features of Field's account, albeit within a more restricted context (in the sense that it doesn't involve a whole scientific theory like classical mechanics). This shows that a measurement-theoretic perspective is adequate to discuss Field's views. In Section 3, I focus, within this perspective, on the explanation of applicability that emerges from the concept of *representation*, which is crucial for the developments of Field 1980. In section 4, I show that such an explanation is not complete and, using ideas presented in Niederée 1992, I supplement it, drawing from the extended account thus obtaining my main argument against instrumentalism. In section 5, I make some concluding remarks to strengthen my position.

2. The applicability of real numbers: Measurement and Field's strategy

Field observes in his book that

> Measurement theory has [focused] on such questions as: what must the intrinsic facts about temperature differences between physical objects be if it is appropriate to think of temperature as being represented by real numbers? And except for the fact that I am substituting space-time points for physical objects, this is in effect the question I am now asking (Field 1980: 58).

Apart from the specific reference to temperature, the point is that the basic question of measurement theory[3] (which, note, is a question about the applicability of real numbers) is exactly the one Field answers in his book. For this reason, and also because the methods adopted to provide an answer are the same, measurement theory can be legitimately used, in the stead of classical mechanics or space-time, to illustrate Field's ideas. This is exactly what I will do in the following sections: nonetheless, in order to better justify my strategy, I'll now illustrate on which grounds we can consider Field's methods and measurement-theoretic ones equivalent.

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entities as approximate descriptions of certain classes of facts, for instance infinite generalizations of them: such generalizations contain elements of objectivity, since they grant the extension of our knowledge, while on the other hand cannot be considered totally objective, in that they are not exhaustive descriptions of reality.

[3] By 'measurement theory' I hereafter intend that mathematical approach to measurement that is also called the representational paradigm of measurement and that has been systematically expounded in Krantz et al. 1971 (two further volumes extending the results in the book exist, but I won't use nor refer to them in this paper). I'm not assuming there exists only one theory of measurement, for many approaches alternative to the representational paradigm have been developed.

One direction of the equivalence (from Field to measurement theory) is trivial for, if we start from Field's book, we see that, to treat quantities like temperature he simply adopts their axiomatizations as found in the works of measurement theorists.

The opposite direction (from measurement theory to Field) requires some observations: what I want to show is that, firstly, the way in which measurement theory treats the applicability of numbers can be rephrased in terms of dispensability and, secondly, that it is possible to easily extend it to encompass the whole domain of Field's analysis.

In order to do it, some preliminary discussion is needed of the way in which physical quantities are treated from the point of view of measurement theory.

One good way of doing it is to start from looking at what is presupposed by our everyday measurement practices: for example, when we need to know how heavy an object is, we simply put it on a balance and read the pointer, thus immediately identifying the sought weight with a number on a scale. Analogously, a chemist that wanted to evaluate the mass of a certain amount of a substance, would use to that purpose a system of standard weights – i.e. such that a number is attached to each of them – and would just sum the numbers of the weights that, on an equal-arm balance, produce a state of equilibrium when put on the pan opposite to that containing the substance whose weight is to be measured. This is to show that we are used to dealing with quantitative attributes of things numerically in a very straightforward way, exactly as when we make calculations in physics. For instance we treat in a purely arithmetical fashion the sum of two dimensional quantities, e.g. we consider the equality $1g + 2g = 2g + 1g$ (where g stands for grams) to be trivially true of masses.

Yet, to see that this may be false, consider the following ideal experimental setting for mass measurement, described by Falmagne 1975:

> In a vacuum room two vertical cables running from the floor to the ceiling have been fixed symmetrically with respect to [a hole] in the ceiling of the room and [an edge] on the floor. The edge is exactly below the hole. […] The experimenter has a collection of homogeneous iron balls which can be hung to the cables (1975:139).

If two spherical balls are hung to the cables, the following procedure is adopted to establish which has the greater mass: a small object is dropped through the hole in the ceiling and one observes on which side of the edge on the floor it falls. This is the side of the heavier object, because of Newton's universal law of gravitation: in other words, the greater mass exerts the stronger attraction.

Within the same setting one could also compare couples of spherical objects (or collections of them, provided they fit in the room), simply hanging each couple to a different cable and repeating the procedure just described. It is apparent that in this sense we are comparing the 'overall' mass of the spherical objects involved, that is their 'sum', so we have an empirical operation of concatenation that we may provisionally think of as additive and that I'll denote simply by '+'. Now, suppose we have four objects *X*, *X'*, *Y* and *Y'*, such that the mass of *X* and *X'* are equal and greater than the equal masses of *Y* and *Y'*: this is also true of $X + Y$ and $X' + Y'$, when *X*, *Y* and *X*, *Y'* are hung to the cable in the same order (i.e. *X* and *X'* come first and closer to the ceiling, *Y* and *Y'* second). Nevertheless, if we hang the two couples in reverse order (i.e. *X* and *Y'* come first, *Y* and *X'* second), the overall masses are, by the measurement procedure, different. This happens because, when an object *O*, dropped through the hole in the ceiling, starts falling, the gravitational pull toward *X* prevails on that in the direction of *Y'* (because *Y'* has a smaller mass) so the distance of the falling object from *X* becomes smaller than that from *Y'*. When *O* continues falling, it therefore finds itself closer to *Y* than to *X*: despite the fact that *X* has the greater mass, it is now farther from *O* than *Y* and, since gravitational attraction decreases non linearly, *X*'s pull on *O* can't cancel out that of *Y* and bring *O* exactly on the edge of the floor. In fact *O* will fall closer to the centre of *Y* than to that of *X*: this shows that $X' + Y' = X + Y \neq Y' + X'$ and the empirical 'sum' defined by the procedure is not commutative. In this case, an equality like $1g + 2g = 2g + 1g$ (assuming we may be incredibly precise, and detect the gravitational forces produced by very small masses on *O*) is false, so we have a situation in which arithmetical laws and empirical ones fall apart.

The main consequence of this example is that there exists a factual distinction between the behaviour of concrete objects subjected to an experimental procedure and the laws of arithmetic: in other words, we cannot apply these laws to any empirical context but only to those that fulfil certain conditions. To put it differently, the applicability of numbers is in effect an 'empirical' matter, in the sense that it depends on the structural features of the objects to be measured.

Looking more closely at the example above, it can be seen that we have there a method for saying which of two objects has the greater mass, i.e. a way of comparing them, and also a method to compare collections of objects, by a concrete operation of concatenation (hanging several balls to one cable). We have, therefore, determined, by means of a procedure, what is usually called in measurement theory an *empirical structure* or *system*, whose domain is the domain of tested spherical balls, and on which an empirical relation of comparison and an empirical operation of concatenation are defined. It has also been shown that the relation of concatenation doesn't behave like arithmetical addition, in the precise sense that it violates one formal property of real addition, namely commutativity: in other words, it is not possible, intuitively, to interpret the empirical structure on the ordered reals with ordinary addition (or a subsystem thereof).

This in turn highlights the fact that, in order to apply that fragment of real arithmetic to an empirical structure, we need certain conditions, concerning the formal properties of the latter, to be fulfilled. We can therefore move one step forward to focus on the difference between the formal properties of an empirical domain and those of arithmetic and conclude, in particular, that the latter only applies to those domains that meet certain factual constraints corresponding to its structural features. We apply, in other words, a certain numerical structure to an empirical one exactly when the latter can be consistently interpreted on the former.

In the case at hand, we start from a precise type of empirical structure, determined by a binary relation of mass-comparison and a binary operation of mass-concatenation (for example that of adding objects on one pan of a balance), whose candidate numerical interpretations are respectively the ordering of the reals according to magnitude and real addition.

More formally, what we are looking for is a structure-preserving mapping (not necessarily one-to-one or onto) from an empirical system into a numerical one, that interprets mass-comparison, denoted by '$\leq_M$', on the numerical relation denoted by '$\leq$', and mass-concatenation, denoted by '$+_M$' on numerical addition, denoted by '$+$'.

In symbols, given the empirical system, $\mathbf{M} = \langle M, \leq_M, +_M \rangle$, where M is a domain of physical objects and the numerical system $\mathbf{R} = \langle R^+, \leq, + \rangle$, where R^+ are the positive reals (or a subset thereof), we want to have a function f, called a *representation*, of **M** into **R**, such that, for any m, m' in M:

i) $m \leq_M m'$ *if and only if* $f(m) \leq f(m')$

ii) $f(m +_M m') = f(m) + f(m')$.

The function f is then called a strong homomorphism of **M** into **R**[4]. What clauses (i) and (ii) say is essentially that all facts concerning mass-comparison and concatenation can be described by numerical facts involving the reals: the arithmetic we use just translates

[4] It would be a homomorphism in the usual algebraic sense if a conditional replaced the biconditional in (i).

numerically the intrinsic facts about masses and it is for this reason that numbers are not necessary to talk about **M**, but rather encode information about it, expressible just in terms of $\leq_M$ and $+_M$[5]. It is also clear that from clause (ii) one can deduce the following chain of equalities: $f(m +_M m') = f(m) + f(m') = f(m') + f(m) = f(m' +_M m)$, which implies that *f* must assign the same measure to $m +_M m'$ and $m' +_M m$, something that contradicts mass-measurement based on the law of gravitation, as described above. In that case no representation on the additive reals could exist, while we can still talk about the empirical structure without one. Thus in some cases numbers *must* be dispensed with (at least as long as we take the structure of **R**) while, more importantly, the problem arises of finding necessary and sufficient, or at least sufficient, conditions for representability (cf. Field's quote at the beginning of this section).

These are usually given in the form of an axiom system from which a metatheorem, called a *representation theorem*, can be proved, ensuring the existence of a function satisfying (i) and (ii) for all the models of the axioms. The axioms here only describe facts concerning physical objects and the theorem shows that, when such facts actually obtain, it is possible to treat them arithmetically: the applicability of numbers is thus subordinated to a concept of empirical regularity (expressed by the axioms), that becomes the criterion to decide when certain numerical systems are adequate to capture the general features of a physical domain. The notion of an adequacy criterion contains the idea that numbers may be dispensed with exactly because they are just assigned to objects through a representation which generates them on the basis of the empirical interactions of the objects to be measured. That is why the logical analysis of measurement in terms of representability is essentially linked to the nominalistic idea of dispensing with numbers and, as a consequence, the same concept of applicability is to be found in both.

If we now think again of order and addition on the reals, but add multiplication and regard them as the basic operations adopted to reflect facts about order and distance (i.e. betweenness and congruence) of points, we see how the representational view can be extended to include geometry and eventually, when the affined geometry of space-time is considered, encompasses the fundamental background of a physical theory like classical mechanics too: the further step, a full treatment of classical mechanics, is completely

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[5] Numbers then are applied in virtue of the conditions on *f*. For instance, using (ii) we can infer from the known values of two masses the value of the mass that will be in equilibrium with their concatenation (simply performing addition).

analogous to the previous ones. Hence the strong connection between measurement, understood in terms of structure-preserving mappings, and Field's nominalism.

3. Representation and applicability

In order to see the relevance of Field's account of applicability, which, as I've shown, essentially coincides with a representational account of the way numbers are assigned to the elements of an empirical system, it is necessary to have a look at what exactly is and what is entailed by a representation theorem. To this end and for future reference, I give below an axiom system from which there can be proved the existence of a mapping that satisfies (i) and (ii) of the previous section.

The axioms I present characterize what are called, in the literature on measurement, extensive systems, i.e. domains of quantities on which a weak ordering[6] and an associative operation of concatenation are defined, and have been introduced in Suppes 1951[7]. Of course, they are adequate for mass-measurement as well as the measurement of almost any other physical quantity[8]. According to Suppes' definition, an extensive system is a triple $\mathbf{M} = <M, \leq_M, +_M>$, where $\leq_M$ is a binary relation and $+_M$ a binary operation satisfying the following axioms[9] (wherein x,y, and z are assumed to be arbitrary elements of M):

1) *Transitivity of* $\leq_M$: $x \leq_M y$ and $y \leq_M z$ imply $x \leq_M z$;
2) *Closure of* $+_M$: if x, y are in M then their concatenation $x +_M y$ is in M;
3) *Weak associativity of* $+_M$: $(x+_M y) +_M z \leq_M x+_M (y +_M z)$;
4) *Weak monotonicity of* $+_M$ *with respect to* $\leq_M$: if $x \leq_M y$, then $x+_M z \leq_M z +_M y$;
5) *Solvability*: if $x\leq_M y$ and not $y \leq_M x$, there exists a z such that $x \leq_M z+_M y$ and $z+_M y \leq_M x$;
6) *Non maximality of* $+_M$ *with respect to* $\leq_M$: not $x \leq_M x+_M y$;

[6] That is a binary relation which is transitive and connected (i.e. intuitively any two elements of the domain on which it is defined are comparable).
[7] Many modifications of this axiomatization, i.e. probabilistic ones or just several weakened forms of it, are at present available. For the purposes of this paper though, and to offer a sufficiently intuitive axiom system, I've decided to use the classical version of the theory proposed by Suppes (which in turn weakens the one of Hölder 1901).
[8] By the fact that they're adequate I mean that the working physicists is generally implicitly assuming them or even, sometimes, a stronger version of them.
[9] I provisionally ask the reader to take this list of axioms without further comments. The justification for having presented them will appear in the next section. For the moment they are used just to give a precise formulation of what is an axiom system for a theory of measurement.

7) *Archimedes*: if $x \leq_M y$, then there is a positive integer n such that not $nx \leq_M y$[10].

It can be shown (for a sketch of the argument see next section), that any model of an extensive system is homomorphic to a subsystem of **R** = $<R^+, <, +>$. Such a result means essentially one thing, i.e. that any model of the axioms can be represented on a subsystem of real arithmetic, whence it is gathered that **R** (or parts of it) suffices to characterize the whole class of models of extensive structures. The general consequence of this result for applicability is that, whenever an empirical domain satisfies the axioms, there is a fragment of arithmetic that can be used to work on it or, equivalently, its elements can be taken to form a metric series on the real line, so that this becomes the general structural feature of all extensive quantities.

Another way of looking at a representation theorem consists in reading it as a solvability condition, to the effect that any empirical domain exhibiting the degree of regularity specified by the axioms gives rise to empirical interactions involving comparisons and concatenations of physical objects that always have a numerical solution on the reals. We might think of such interactions concretely as the outcomes of an experimental procedure applied to a certain physical domain and more abstractly as a class of conditions emerging from the procedure (and ultimately reducible to the axioms): thus a procedure generates a sort of empirical set of 'constraint equations' in several unknowns, that, as long as they reflect the features of an extensive domain, can always be solved numerically, i.e. replacing the unknowns with numbers (under a suitable interpretation of the operations and relations involved).

One way in which Field's nominalism is relevant for the description of applicability is its stress on representability as solvability, as just outlined: that such a stress is really present in Field's discussion can be seen indirectly, looking at one aspect of solvability whose importance he explicitly acknowledges. Strictly speaking, what a representation theorem tells us is that, if something is a model of the Suppes' axioms, then there is at least one function that maps it on the additive reals. So we have at least one numerical solution of the axioms and this naturally poses the question how many of them there are. If we look at extensive systems, we can quite clearly see that there are infinitely many numerical solutions, that can

[10] Here the integer multiples of a quantity x are defined recursively by means of concatenation, i.e. by putting $1x = x$ and $(n+1)x = nx +_M x$. The connectedness of the ordering, as well as the commutativity of $+_M$ can be derived from the axioms (the particular form of (4) is introduced to facilitate the derivation of commutativity). Also, (3) doesn't assume full associativity (that would be given by (3) plus the opposite inequality, since this is the way in which empirical equality $=_M$ can be defined from a weak order), but this follows from the axioms anyway. Algebraically, an extensive system of the type described can be thought of as an ordered, positive and Archimedean semigroup.

be identified with those one-to-one transformations of the reals onto themselves which leave their ordered additive structure unchanged (i.e. the order-preserving automorphisms of the additive reals). In symbols, if *f* is a representation and *g* one of the relevant transformations of the reals, then *gf* (the functional composition of *f* and *g*) is still a representation, and it can be proved that nothing else is a representation of an extensive structure on the ordered, additive reals. Such a result is called a *uniqueness theorem* and gives information about how strongly the empirical structure of a given domain constrains its numerical interpretations[11]: it turns out that, in the case of extensive systems, the *g*'s correspond to the class of similarity transformations, i.e. multiplications by a positive real constant. Thus one could take the values of *f* and multiply them for one fixed positive real number, to obtain a new representation from *f*: the numerical transformation thus performed corresponds to an empirical change in the unit of measure (for Field's remarks on uniqueness results, see Field 1980: 50).

From this point of view, representation and uniqueness theorems correspond just to existence and uniqueness constraints induced by an empirical structure on its possible numerical representations. It could be shown (see Krantz et al.1971: 99-102) that, even if we did not represent extensive structures on the additive reals but chose to interpret them on some other isomorphic numerical structure (for instance one where we keep the same order relation but replace ordinary addition with ordinary multiplication), we could straightforwardly obtain from the additive case representation and uniqueness results for the other representing structures. All these alternative cases, in which we obtain the existence of a determined class of morphisms connecting a physical system to an infinite class of numerical structures, are unified by the axioms for the physical system concerned, which in every instance require the identity of structure of the representing systems and their having an associated group of transformations which vary with one degree of freedom, corresponding to the concrete act of changing the unit of measure.

Because, finally, the numerical solutions of an axiom system can be generated by a structure-preserving function, it is quite clear that they are introduced exactly to keep track of the outcomes of concrete operations performed on an empirical system, which are left undisturbed under changes of unit. In other words, to each empirical comparison of, say, masses there corresponds a numerical inequality and to each concatenation of masses a

[11] An interesting related question is the converse one, asking how strongly certain uniqueness properties constrain the structural features of a domain which is required to entail them.

numerical sum: performing sums and studying inequalities becomes a way of extracting a numerical algebra from a procedure, which crucially passes through axiomatization. In this context we understand numbers as a way of singling out certain salient features of an empirical domain and, also, of making systematic deductions about it.

The main result we achieve, and that agrees with Field's discussion of representation theorems in Field 1980, consists in detaching numbers from empirical facts and showing how the two interact: we thus free ourselves from any form of holism that regards numbers as entities inextricably entangled with the scientific analysis of physical facts. On the contrary, we are in a position (here only restricted to measurement, but generalizable to full physical theories) to separate intrinsic facts about quantities from extrinsic facts about numbers and this conclusion is certainly in line with Field's nominalism[12] .

On the other hand, *the theory of real numbers* […] *was developed precisely in order to deal with physical space and physical time and various theories in which space and/or time play an important role, such as Newtonian mechanics* (Field 1980: 33), so we really don't want to stop at the dichotomy between intrinsic and extrinsic facts but understand, at a deeper level, in which sense the latter can be considered as *schemata* for the former. In other words, we need to remove ourselves from a perspective which exclusively understands the application of real numbers as the solving of empirical constraints, in order to see in which sense it is possible to see the real numbers arise from 'experience'. In order to do this, we need to get back to the representation theorem and try to look more closely at its structure, in order to isolate another fundamental feature of measurement, that should be coupled with solvability, namely *assignment* or *evaluation*. A beautiful consequence of this shift of perspective is that it yields a nominalistically acceptable treatment not only of quantities, but also of their measures.

4. Measurement without numbers

Let us look at the way Suppes proves the representation theorem for extensive systems from his axioms (see beginning of previous section). First of all let us note that axioms (2) and (6)

Field writes:

I am saying that not only is it much likely that we can eliminate numbers from science […] but also that it is more illuminating to do so. It is more illuminating because the elimination of numbers helps us to further a plausible methodological principle: the principle that underlying every good extrinsic explanation there is an intrinsic explanation (Field 1980: 44).

imply the existence of infinitely large elements, simply because $x +_M x$ exists by (2) and is strictly bigger than x, by (6), so $(x +_M x) +_M x$ exists and is strictly bigger than $x +_M x$ and so on *ad infinitum*. We have therefore unbounded sequences of multiples of any element of M (such sequences are uniquely determined by associativity ((3) plus its converse), for without it we should distinguish them on the basis of the way they have been generated and fix a uniform method of generation). On the other hand, by (7) we know that, whenever we fix any two elements of M, call them m and m', there always is one multiple of the smaller that exceeds the bigger. Furthermore, it follows from (7) that, if $m \leq_M m'$, then there is an integer k such that $km \leq_M m' <_M (k+1)m$ so[13] m' can always be bracketed within an interval that occurs at some point of the infinite sequence of the multiples of m: since such sequence is unbounded, this will happen for any m' which is bigger than m. Now we can choose m as unit of measure and determine m' in terms of m with any desired precision: clearly, since we cannot reduce the precision by dividing m up into parts, we have to go the opposite way, and take bigger and bigger multiples of m, whose existence is granted, to be compared with multiples of m' . The reason why this is done can be immediately illustrated by observing that, if m' is 12.333 times m, this is equivalent to saying that 12333 copies of m' will equal 1000 copies of m: instead of talking directly about thousandths of the unit, we use the thousandth multiple of it. The equivalence depends on the fact that the ratio of m to m' is, in both cases, the same, i.e. 12.333. Now the strategy to get a representation theorem from Suppes' axioms is, strictly speaking, all contained in the observations just made: what it exploits is the fact that, using the properties of an extensive structure, we can construct unbounded standard sequences, i.e. sequences of multiples of any given element of M, that eventually will reach and exceed any other, bracketing it within intervals that, in the sense just explained, can be made to correspond to any arbitrary precision. Clearly, we do not take only multiples of the unit of measure m but also multiples of the objects compared against it and consider the inequalities or equalities arising from comparing such multiples. Using the above example, for instance, we get the empirical equality $12333m =_M 1000m'$, and we would also naturally get, for instance, $12300m <_M 1000m'$ and $12300m >_M 500m'$: by trichotomy, which is an immediate property of the empirical ordering, all empirical comparisons must have the form of exactly one of the three cases just exemplified. Thus we may divide the outcomes of all possible empirical comparisons into three classes, which will be respectively determined

[13] Note that $m >_M m'$ is defined as $\neg(m \leq_M m')$.

by the fractions $\frac{k}{h}$ such that k$m <_M$ hm'or k$m >_M$ km' or k$m =_M$ hm' , where k,h are positive integers. The reader familiar with Dedekind's construction of the real numbers will have noticed that the three classes of fractions thus defined determine a cut on the real numbers and that this cut will be an irrational number when the third class is empty[14]. In other words, if we imagine to be working on the reals, we can determine any real number *r* using a partition of all the rationals $\frac{k}{h}$ into those that are smaller than *r*, those that are bigger than it and those that equal it: strictly speaking, and also exploiting the fact that we are dealing with positive quantities, *r* can be uniquely determined by the positive rationals that are smaller than it (thanks to the fact that the rationals are dense in the reals).

If we consider all inequalities of the form k$m <_M$ hm' and call their associated fractions a *lower cut*, then we can prove from Suppes' axioms that such lower cut hasn't got a maximal element, and therefore contains countably many fractions that form an increasing sequence: this sequence has always a limit in the reals (because of the property called Dedekind completeness) and such a limit is taken to be the measure of *m'* with respect to unit *m*. The function that, once a unit *m* is fixed, assigns to any *m'* the limit of its lower cut is a representation of an extensive system into the additive, positive reals: this is because the linearity of limits with respect to addition is just condition (ii) and because, if $m' <_M m''$ then the lower cut determined by *m'* is included in that determined by *m''*, and set-theoretical inclusion is an order relation fulfilling (i)[15].

What has emerged so far is that the representation theorem for extensive measurement that Suppes gives is strongly connected to the mathematical idealization of a concrete procedure of measurement, that of comparing multiples of quantities, which I'll call the *method of lower cuts*: since such comparisons generates any, however refined, approximation of the ratio of a quantity to the chosen unit of measure, real numbers are naturally introduced as the ideal terminations of such approximations.

It is noteworthy that if, instead of (7), we assumed second-order Dedekind completeness, as Field does for lines in space-time, we would then get limits directly within the empirical structure: the point is that, in a sense, to assume Dedekind completeness in its

[14] This way of describing the numerical measures of quantities was firstly proposed in Hölder 1901.

[15] The strong morphism condition is obtained by restricting oneself to the subsystem of the reals determined by the image of the empirical domain with respect to the representation. Of course this latter won't be one-to-one, because there may be quantities that are equal, and therefore generate the same cut. This is the main reason why in measurement one talks about homomorphisms rather than isomorphisms between structures.

'nominalistic' version and to use the reals as uniquely determined measures of a quantitative domain is really the same thing. On the one hand we postulate completeness as a feature of the reals, on the other we postulate the existence of ideal quantities that provide the limits to our sequences. Clearly the reason why the reals, as Field himself remarks, have been developed to treat motion in classical physics, i.e. to describe such concepts as instantaneous velocity, which require the notion of limit, was exactly because they provided an abstract scheme for the idea of continuous variation and, at the level of measurement, their importance lies in the fact that they provide an abstract schema for the idea of 'true value' of a quantity, which improves on any possible approximation. Yet in both cases we're talking about ideal elements, whose employment is justified not simply by reasons concerning computational convenience, but also by the kind of empirical setting they were designed to capture. To assume this standpoint doesn't seem to me to imply an alignment with the platonistic positions Field rejects, exactly because I fully accept his account of applicability: on the other hand it seems to me that the distinction between strong structural assumptions of a nominalistic kind – those constituting the ontological content of nominalism – and mathematical assumptions becomes very blurry when we take into account the schematic use of mathematical entities (I'll get back to this point in the last section).

Such schematic use, in turn, can be thought of as performing one fundamental function, besides that of using numbers to describe physical facts: this is to assign to each quantity a position within a reference frame, and to present the class of positions thus obtained as an abstract object, encoding the salient features of a quantitative domain. In order to clarify this point, I need to introduce a method to 'nominalize' the measures of quantities, which is nothing but the technique developed in Niederée 1992[16].

The basic idea is the following. Suppose there is an ideal[17] experimenter that carries out comparisons and concatenations on objects of the domain *M*: what he essentially does is to test whether certain quantities are equal or not and, in the latter case, which one is bigger or smaller. Let us also suppose that the experimenter records all the outcomes of the tests he performs, but that he does it in a first-order language *L* containing, apart from all the usual logical symbols, countably many variables and only two non-logical constants, denoting

[16] Although heavily restricted to the context at hand, for Niederée's results are far more general (and equally interesting in both their mathematical and philosophical implications).

[17] I talk about an ideal experimenter because I'm assuming here that he may be able to perform and record infinitely many operations.

respectively the binary relation $\leq_M$ and the binary operation $+_M$[18]. After the experimenter tests two elements of *M*, say *m* and *m'*, he records the infinitely many outcomes of his tests in the form of a list, that I'll call an *empirical record*[19], of atomic formulas of *L* like k$x <_M$ hy, where kx and hy are only abbreviations for the k-th iteration of concatenation (here thought of as a function involved in term-formation) on *x* and the h-th iteration of concatenation on *y*, so that no numerals are actually used. We may assume that the experimenter is only interested in inequalities of the specified kind, and disregards their converses as well as equalities.

We can thus think of an extensive system **M** as a model for atomic formulas like k$x <_M$ hy and say, if in the system k$m <_M$ hm' holds that k$x <_M$ hy is true in **M** under the assignment which maps *x* and *y* into *m* and *m'* respectively.

Suppose in particular that *m* has been chosen as unit of measure and we restrict ourselves to atomic formulas wherein variable *x* occurs, forming a class *A*, and to assignments that map *x* into *m* only. It is clear that, calling *a* any such assignment, the formulas of *A* that are all simultaneously satisfied under an assignment *a*, that maps *y* into *m'* , are all and only those atomic formulas in the empirical record of *m'*, when *m* is chosen as unit (of course, if *m''* is equivalent to *m'*, i.e. if they have the same mass, in the case of mass-measurement, then they identify the same empirical record). In particular, if it is the case that $m' <_M m''$, then their empirical records will differ (for they correspond to two lower cuts, one strictly included in the other), which is to say that there is at least one atomic formula in the empirical record of *m''* which doesn't occur in the empirical record of *m'*: we see, in other words, that empirical records are subsets of *A*, i.e. elements of its powerset *P(A)* that *separate* the elements of *M*, in the sense that, to non-equivalent elements of *M* different elements of *P(A)* are associated, while only one subset of *A* is associated to any element of *M*.

This last remark shows that there is a function (not necessarily one-to-one) from the elements of *M* into their empirical records and it is easy to see that set-theoretical inclusion between elements of *P(A)* corresponds to order on elements of *M* while, using that same function from *M* into *P(A)*, we can also induce on *P(A)* an operation behaving like concatenation on *M*[20]. But in this way we have just obtained a structure-preserving mapping

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[18] In what follows I'll use, just for convenience, the same symbols to denote both the relation and operation defined on an extensive domain and the symbols that denote them.

[19] Niederée 1992: 248 talks in this case about complete records. I preferred to use the adjective 'empirical', although it is strictly speaking not very appropriate, just to stress that we are talking about classes of statements with an intrinsic meaning, that ultimately refers to concrete objects.

[20] If the function in question is *f*, it is sufficient to define an operation $+_{P(A)}$ such that $f(m) +_{P(A)} f(m') = f(m +_M m')$.

from an extensive system into the domain of its associated empirical records, that is, a new representation theorem: in model-theoretical terms, empirical records are just the types of elements of *M* with respect to set *A* and unit *m*.

The interesting fact is that we're now measuring quantities on the empirical records describing their behaviour: the reason why such measures are nominalistically acceptable is that they are not abstract entities like numbers, but simply inscriptions, though idealized ones (as we must have infinitely many of them). The reason why, on the other hand, such measures are epistemologically interesting is that each of them is a kind of complete report of the way any object in *M* behaves with respect to a unit under a fixed procedure involving standard sequences. Thus measures here are but descriptions of the position of an object within a quantitative domain with respect to a unit, which fixes a kind of reference frame.

In the numerical case, what I focused upon discussing the representation theorem was the idea of solving empirical constraints numerically; here, on the other hand, the representation theorem underlines another aspect of measurement, namely the fact that it locates the position of empirical objects within a space of records determined by a procedure. A full empirical record tells us not only how an object behaves with respect to a unit but, at least indirectly, it tells us why it differs from any other (non-equivalent) object in that domain: this is what I call the *evaluation* of a quantity[21] and, as it can be seen, it is an essentially relational process, because it is uniquely determined by an infinity of interactions between an object and a unit of measure.

It's now straightforward to see how evaluation and solvability are connected together, for there's a natural way of associating to any empirical record a lower cut, that has a limit in the reals. Using this fact, is also possible to see exactly how the reals can be regarded as schematic versions of empirical records: as limits of lower cuts they 'encode' the sequence of all approximations that tend to them. For this reason, to apply the reals means to use a general scheme which reduces empirical records to positions along a continuum, i.e. presents a class of positions within a reference frame as an abstract object: reasoning is facilitated by such a scheme, and this exactly because of its deep connection with empirical records. Such justification of the applicability of the reals is better than one stressing the solvability-aspect of a representation theorem only, exactly because it takes into account the evaluation-aspect, makes apparent that real numbers can be considered indices of sequences of approximations, and shows a sense in which they may be said to possess some degree of intrinsic content (to

[21] On this point compare Niederée 1992: 252, on a structure being *representable through values of measurement*.

the extent approximations correspond to empirical records). In this sense it is clear that real numbers emerge as an idealization of an empirical procedure and cannot be considered purely extrinsic to it (for similar reasons differentiable functions emerge from a certain ideal notion of variation, for instance of position through time). In other words, since the concept of an extensive system axiomatized as above is essentially based on the idea that we measure by forming standard sequences and comparing their finite segments, and since the notion of real number emerges exactly from classes of such comparisons, it must be concluded that it is strictly connected to the (indispensable) concept of extensive quantity and cannot be reduced to simply a tool that is invoked from the outside in order to deal with it. This conclusion is presumably in line with Field's anti-realism, but not with his instrumentalism: indeed it calls for revising the contrast between nominalistically acceptable entities and mathematical ones.

5. Epistemological conclusions

I take it that I have shown that the reals, in the case of their application to extensive quantities, play a schematic role in providing a perfectly general series of labels, whose order and metric structure extract the salient features of a physical domain, that are revealed through a physically specifiable measurement procedure. The completeness of the reals, within this perspective, provides us with an abstract object (the ordered series of the reals, but more generally a continuum) that displays, as its elements, all the possible empirical records arising from applying the method of standard sequences with arbitrarily large multiples. Indeed the axioms for an extensive structure, while sufficient but not necessary conditions to interpret numerically an ordered algebra[22] on the additive reals, are necessary *and* sufficient for numerical interpretation *carried out through the method of lower cuts*, which actually is a strong form of the method of standard sequences[23].

To put it more clearly, the axioms reflect exactly the kind of empirical information that is determined by the reals when thought of as limits of approximations: from this standpoint there appear some significant similarities between the axiomatic characterization of extensive quantities and the structure of the reals, conceived as cuts of rationals. It is true that on the one hand we make assumptions about theoretical entities, that are nominalistically acceptable, and on the other hand we talk about numerical entities, that are not, but on both

[22] That is, a set on which an order relation and a binary operation are defined.

[23] In general, such a method can be carried out even with weaker axioms, for instance restricting closure: but in that case we are not allowed, as we just did in the previous section, to form arbitrarily large multiples of any element of a domain, because they might not be defined. So it is probably better to say that Suppes' axioms are necessary and sufficient to apply some strong version of the method of standard sequences.

sides we make those assumptions for the same schematic reason, which is to formally capture a measurement procedure, whose mathematical generalization is just the method of lower cuts itself. We have therefore an interaction between a concrete procedure, its idealized version, axiomatically formulated, and the arithmetic needed to portray it numerically and clearly the features of the procedure give the common conceptual core of its synthetic and analytic depiction.

We work on quantitative data with an idealized model (an extensive system), that yields a highly regular version of them, on which we can argue in mathematical form, i.e. using proofs: on the other hand a representation theorem transfers our reasonings to a numerical domain, but one of the reasons why this latter domain preserves the information we get from the formal model is that its elements can be thought of as generated by infinite sequences of empirical comparisons and, as limits of them, concisely contain the notion of approximation which is involved in a comparison procedure.

These conclusions of course do not imply that we should naively press for a sort of identification of intrinsic entities with numerical ones, under the general category of ideal entities, because if we did that, we would lose, together with our analysis of applicability, the ability to determine, given the primitive notions that we want to use in order to describe a certain class of phenomena, how much arithmetic is needed in order to treat that class numerically. For example, when we coordinatize the points on a line we don't need to invoke the full field-structure of the reals nor indeed their metric structure, as there is not, in this case, an additive operation between points of a line. In general, numerical operations are only satisfactorily analysed into their empirical components when a distinction between intrinsic content and representing structure is kept in mind and this has, in addition, several consequences for a notion of objective content of a theory, which, for example, plays a definite role in the applications of measurement[24].

It is, on the other hand, noteworthy that the choice of a representing numerical structure and the choice of the axioms that entail the representation theorem are sometimes closely intertwined: in the case of extensive quantities a real representation is strongly forced

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[24] One problem, that has arisen especially in economics and psychology, was that of finding a way to discriminate the empirically meaningful results among those determined by the manipulation of numerical values obtained through measurement and reflecting, for instance, ordering of preferences or of subjective sensations. What is debated is whether the application of certain numerical functions, i.e. statistics, to measured values, yields empirical information about the quantities that are measured (criteria to decide whether this is the case have been developed, on the basis of the invariance properties of measurement scales obtained through uniqueness theorems). Of course such a problem involves a distinction between those numerical facts that haven't got an empirical content and those that have.

by the introduction of the Archimedean axiom (7). This happens because Archimedes' governs the finite comparability of quantities and so plays a crucial role for the possibility of constructing approximations with the method of lower cuts, while on the other hand some version of archimedeanity is always entailed by the existence of a real representation (not by chance, obviously, but because of the empirical intuition underlying the construction of the reals). However, archimedeanity is empirically necessary only insofar as we fix an empirical procedure (involving unbounded standard sequences) based on a precise concept of approximation, but it should be observed that there exist other measurement procedures, on the basis of which successive approximations might not play a particularly significant role. If we therefore decided to change procedure, axiom (7) might become a purely theoretical condition, that were simply adopted to fix a representing structure, namely the standard reals (instead of, say, the non-standard reals). To make this remark less generic, an alternative concrete measuring procedure[25] for extensive quantities, well-known to measurement theorists, may be sketched here: we start from atomic formulas like the inequalities of previous sections, but this time without any restriction on the occurrences of variables and also allowing for weak inequalities; we, nevertheless, only take into account finite lists of inequalities, denoting empirical comparisons or data; to represent an extensive system thus becomes to find real numbers that simultaneously solve our finite set of inequalities or, equivalently, satisfy a finite set of data.

It is quite clear that no notion of approximation is involved, the same uniqueness theorem for extensive quantities continues to hold and, finally, we use the more realistic condition of restricting ourselves only to finitely many atomic formulas. It has been shown by Adams et al.(1970) that, on the basis of this measurement procedure, a system of data of the specified kind[26] is solvable on the reals if and only if it satisfies axioms (1) to (6), i.e. independently of Archimedes. This clearly shows that changes in the measurement procedure induce changes in the empirical content of the axioms and, in particular, that, on the basis of this fact, they allow for a larger freedom in the choice of representing structure. In the case just mentioned we might either add the Archimedean axiom and fix the usual notion of approximation or, using the elementary equivalence of first-order models of the reals, which ensures that the same system of data (first-order formulas) has solutions on the non-standard reals as well, find a representation on a non Archimedean domain, thereby changing our

[25] Here I take 'measuring procedure' and ' method to establish a representation' to be synonymous.
[26] Where data here are finite segments of empirical records or their combinations.

notion of approximation, that now allows for intrinsic infinitesimal fluctuations of quantities (something that doesn't appear particularly extravagant, having in mind e.g. uncertainty principles).

Here again we see that, under changes of measuring procedure, the relationship between axioms and representing structures becomes slightly more complex than in the cases examined in the previous section: here the choice of a numerical structure carries with itself a kind of theoretical presupposition concerning the notion of approximation involved and, ultimately, the concept of quantity that one extrapolates to. The point of this discussion is to stress that, on the one hand, different representing structures possess different evaluative features, i.e. their formal properties are made to correspond to certain general features of an empirical domain, while on the other hand they generalize and extend what is given through experiment. We see that there is a non trivial interplay between an empirical structure and the way it may be represented, numerically or non numerically, due to the fact that the structure whereon a given empirical domain is interpreted provides a universal characterization of the former. What I mean by this expression may be explained with reference, again, to the case of real measures for extensive quantities: when they are introduced, they appear as limits of ideally infinite sequences of approximations and it is thus the completeness of the reals, i.e. a topological property of a numerical structure, to express the empirical behaviour of the concrete notion of approximation of a quantity with respect to a unit by means of successive comparisons of multiples. Completeness says that any sequence of numerically determined approximations (which is increasing and bounded above) has a least upper bound or, equivalently, that it converges to a uniquely determined value. This is in fact an axiom of the reals, but it plays a crucial role in the representation theorem for extensive magnitudes because it says what would eventually happen if we could indefinitely refine the accuracy of a measurement procedure. Obviously we are never in a position to test such a possible outcome: the importance of stating it though doesn't lie in its testability but rather in the possibility of giving a uniquely determined description of extensive quantities as a subsystem of an Archimedean continuum. The formal structure of measures, i.e. the formal structure of the reals, provides then a general conceptual framework within which it is possible to understand the behaviour of quantities: this framework goes beyond experience and it is only in this way that we can give a completely determined description of quantities, because through experience only we would be forced to stop at finite truncations of potentially infinite sequences of approximations. Here it is clearly evinced that the reals play a schematic role in the theory of extensive quantities, in the sense that they provide a systematic way of fixing their interrelationships, which builds on experience (the concrete fact that we can improve the accuracy of a measurement procedure) but goes beyond it, a necessary condition to 'complete' or 'close' the limited information which is experimentally available. Therefore real numbers appear as an extrapolation from the structural features of empirical domains of extensive quantities[27]: this perspective has some important philosophical consequences. One of them is that it gives an account of applicability that clearly avoids and improves on Field's view about the extrinsicity of representing structures: the reason why we apply numbers to objects depends on the fact that the former provide an objective and universal characterization of the latter or, equivalently, from a twofold requirement of compatibility with experience and generality.

More precisely, the first requirement is satisfied by the fact that the (ordered, positive) real numbers are a model of Suppes' axioms, i.e. they obey the empirical constraints spelled out in the axiom system, while the second is satisfied by the fact that any model of Suppes' axioms finds a uniquely determined evaluation for its elements (once a unit of measure has been fixed) on the reals. Like Field, I do not consider numbers as independently existing entities that prove essential to the treatment of extensive quantities but I think it particularly significant to see them as a conceptual framework emerging from an analysis of experience coupled to a generalization from experience.

If instead of the real numbers directly an axiom system for them is considered, the previous observations may be rephrased in terms of the interrelationships between a set of constraints for quantities, e.g. Suppes' axiom system, and a set of constraints for continua with an additive structure on them, i.e. an axiom system for complete[28] ordered semigroups, amongst which the additive reals are to be counted. In terms of solvability, we may say that any solution to the first set of constraints can be associated with a system of labels satisfying the second set of constraints, while in terms of evaluation we may say that the second set of constraints is such that its solutions reflect all the possible interactions between the empirical elements satisfying the first set[29]. Thus by the schematic role played by the reals in extensive

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[27] It has to be noted that Suppes' axioms are in effect already an extrapolation, since their models are forced to be infinite.

[28] Where, if an Archimedean condition occurs among the axioms for an extensive structure, completeness is thought of as being formulated as an axiom rather than an infinite scheme of axioms.

[29] Note that this view can be generalized to other theories of measurement and it is by no means restricted to extensive quantities. Any other case could be treated exactly as the one I have considered, with the sole exception of considering types of structure other than the one described in this paper.

measurement I simply mean the kind of interaction between sets of constraints just described: the schemas concerned are, in this case, axiom systems[30].

The analysis of the schematic role played by mathematical objects thus developed makes it possible to draw two conclusions concerning Field's nominalism. The first one, whereon I indirectly already insisted upon, is that the instrumentalist position coupled with his nominalism should be rejected on the basis of the evaluative role of numerical entities. Incidentally, instrumentalism based on the concept of extrinsicity seems to have much in common with mathematical realism, in that it takes for granted (or at least it can well do it) the standard realist view on numbers and rejects it only insofar as it can rely on a way for dispensing with numbers. From the standpoint I have articulated in this paper, such a form of instrumentalism is unacceptable because it fails to acknowledge the actual interaction between mathematical and empirical structures and the way in which the first ones may be seen to emerge as general theoretical hypotheses or extrapolations concerning the second ones.

The second conclusion concerning Field's nominalism is relative to his assertion that

> Postulating uncountably many physical entities […] is not an objection to nominalism; nor does it become more objectionable when one postulates that these physical entities obey structural assumptions analogous to the ones that Platonists postulate for the real numbers (Field 1980: 31).

This statement could be easily reread as an assertion of the evaluative status of mathematical objects, because it stresses the fact that we construct models for empirical phenomena using those structural constraints that are usually imposed upon numerical entities, which in turn depends upon the fact that numerical entities themselves can be considered idealization of concrete procedures, as we have seen in the case of additive quantities.

At the same time, the fact that the nominalist should ontologically commit itself to the existence of entities that he postulates through extrapolations (e.g. the fact that there are uncountably many entities) seems too strong and unsatisfactory in the light of, for instance, the results of Adams et al. (1970) mentioned above, which clearly show the possibility of different generalization from the same class of empirical data. Again, taken at face value,

[30] Now the previously discussed example taken from Adams et al. (1970) shows that, if the set of empirical constraints does not contain Archimedes' axiom, we can make it interact with a first-order axiomatization of complete ordered semi-groups, which allows for the possibility of non Archimedean measures. This means that, for a class of empirical interactions within an extensive domain, we are in a position to extrapolate from it in different ways, thereby obtaining different general notions of extensive quantity: the same outcome would occur if we assumed first-order completeness as an empirical axiom.

Field's nominalism exhibits some form of isomorphism to mathematical realism, with the difference that it rephrases its claims in intrinsic terms, because it doesn't mention the fact that the intrinsic assumptions upon which it is based go beyond direct experimental evidence. It is clear that this is not one of Field's concerns, and yet it should be if the analysis of applicability that he implicitly provides is to be taken seriously, while it seems advisable for his nominalism because it would supplement it with a critical assessment of certain strong assumptions made on synthetic models of physical settings. We may regard these assumptions as constraints with a heuristic or theoretical value which direct our ways of reasoning on certain empirical facts, without thereby straightforwardly committing ourselves to their ontological import, as long as we see constraints as axioms and the relation of logical consequence as directing the way in which we articulate the heuristic or theoretical implications of the axioms (in which case it becomes a problem to have, as happens in Field 1980, a notion of logical consequence which is not recursively axiomatizable). Such conclusions are quite natural from the schematic viewpoint I have presented, if we think, as suggested above, of schemas as sets of constraints or axioms (cf. the previous analysis of the role played by the completeness of real numbers with respect to extensive quantities, which shows how a continuum can generalize an additive structure[31]: here we start from consequences of the empirical axioms, that describe sequences of approximations, and look at their interaction with the completeness axiom for the reals, which establishes a representation theorem).

To conclude, I think it is fruitful, on the basis of the schematic account of applicability I have proposed, to modify Field's view by essentially dropping its instrumentalistic implications and reconsidering the role of strong structural assumptions on empirical settings, in order to obtain a deeper understanding of the way we use numbers (and, more generally, mathematical theories) to talk about facts.

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[31] Similar observations could be extended to other strong assumptions on extensive quantities, like closure with respect to concatenation, by examining axiom systems that do not satisfy them and yet are representable on the additive reals.

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